OR RefCard II 2011

1 Cadenas de Markov en Tiempo Continuo

1. Definiciones

   (a) $X(t), t \geq 0$, con S-estados es CMTC:
   $P(X(t) = j|X(s) = i, X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1) = P(X(t) = j|X(s) = i)$

   (b) CMTC homogénea:
   $P(X(t) = j|X(s) = i) = P(X(t-s) = j|X(0) = i), \forall s \leq t,$
   $\forall i, j \in S$

   (c) $T_i \sim Exp(v_i)$, tiempo de permanencia en el estado i.

   (d) $X(t), t \geq 0$ es CMTC: los tiempos de permanencia son Exp y las transiciones son según $p_{ij}, p_{ii} = 0, \sum p_{ij} = 1$

   (e) Poisson: $p_{i,i+1} = 1, p_{ij} = 0, \forall j \neq i+1$ y $v_i = \lambda, \forall i$

   (f) Yule: $p_{i,i+1} = 1, p_{ij} = 0, \forall j \neq i+1$ y $v_i = i\lambda, \forall i$

   (g) 2-procs (μ_1, μ_2) secuenciales y llegadas $P(\lambda)$:
   estados {00/0,10/1,01/2,11/3}
   $T_0 \sim Exp(\lambda)$, $T_1 \sim Exp(\mu_1)$, $T_2 \sim Exp(\lambda + \mu_2)$, $T_3 \sim Exp(\mu_2)$
   $p_{01} = p_{12} = 1$, $p_{20} = \frac{\mu_2}{\lambda+\mu_2}$, $p_{23} = \frac{\lambda}{\lambda+\mu_2}$, $p_{31} = \frac{\mu_2}{\mu_1+\mu_2}$, $p_{32} = \frac{\mu_1}{\mu_1+\mu_2}$, resto 0.

2. Comportamiento de Transición

   (a) $p_{ij}(t) = P(X(t) = j|X(0) = i)$

   (b) Ec. Chapman-Kolmogorov:
   $P_{ij}(t+s) = \sum_{k \in S} p_{ik}(t) p_{kj}(s)$ (hacia atrás)

   (c) Tasas instantáneas de transición:
   $q_{ij} = v_i p_{ij}$, $v_i = \sum_j q_{ij}, p_{ij} = \frac{p_{ij}}{\sum_j q_{ij}}$

   (d) EcDif de p_{ij}: $p'_{ij} = \sum_{k \neq j}^{\infty} p_{ik}(t) p_{kj} - p_{ij}(t) v_j$, $\forall i, j, t \geq 0$

   (e) Poisson: $p'_{ij} = p_{i,j-1}(t)\lambda - p_{ij}\lambda$,
   $q_{i,i+1} = v_i p_{i,i+1} = v_i = \lambda$, $q_{ij} = v_i p_{ij} = 0, \forall j \neq i+1$

   (f) Yule: $p'_{ij} = p_{i,j-1}(t)(j-1)\lambda - p_{ij} j\lambda$,
   $q_{i,i+1} = v_i p_{i,i+1} = i\lambda$, $q_{ij} = v_i p_{ij} = 0, \forall j \neq i+1$

   (g) 2-procs: $q_{01} = \lambda$, $q_{12} = \mu_1$, $q_{20} = \mu_2$, $q_{23} = \lambda$, $q_{31} = \mu_2$, $q_{32} = \mu_1$, $\forall j \neq i+1$, $p'_{00} = p_{02}(t)\mu_2 - p_{00}(t)\lambda$,
   $p'_{01} = p_{00}(t)\lambda + p_{03}(t)\mu_2 - p_{01}(t)\mu_1$

   (h) $P'(t) = GP(t)$, $g_{ii} = -v_i, g_{ij} = q_{ij}, i \neq j$, G generador, $P(t) = e^{tG}$

   (i) Diagrama de transición: nodos-estados, etiqueta del arco ij q_{ij}

3. Comportamiento Estacionario

(a) Distribución estacionaria:
$\pi = \pi P(t), \forall t \geq 0 \to 0 = \pi G, \sum pi = 1$

(b) En CMTC si $\exists \pi$, es única,

   i. Ecuaciones equilibrio globales (EEG):
   $0 = \pi G, \sum pi = 1$

   ii. Ecuaciones equilibrio locales (EEL):
   $\pi_i q_{ij} = \pi_j q_{ji}, \forall i \neq j \in S$, CMTC reversible

(c) Poisson: $\lambda \pi_j = \lambda \pi_{j-1}, \forall j, \sum pi = 1$

(d) Yule: $j\lambda \pi_j = (j-1)\lambda \pi_{j-1}, \forall j, \sum pi = 1$

(e) 2-procs: $\lambda \pi_0 = \mu_2 \pi_2$, $\mu_1 \pi_1 = \lambda \pi_0 + \frac{\mu_2^2 \pi_3}{\mu_1+\mu_2}$,
$(\mu_1+\mu_2)\pi_2 = \mu_1 \pi_1 + \frac{\mu_1\mu_2\pi_3}{\mu_1+\mu_2}$, $\frac{\mu_2^2+\mu_1\mu_2}{\mu_1+\mu_2}\pi_3$, $\sum pi = 1$

4. Procesos de Nacimiento y Muerte (PNM)

(a) PNM con tasas de nacimiento $\lambda_i, i = 0, 1, \ldots, \infty$ y tasa de muerte $\mu_j, j = 1, \ldots, \infty$ es un CMTC, $S = \{0, 1, 2, \ldots, \infty\}$,
tasas de permanencia $v_0 = \lambda_0, v_i = \lambda_i + \mu_i, i > 0$, y
$p_{01} = 1, p_{0i} = 0, i \neq 1$,
$p_{i,i+1} = \frac{\lambda_i}{\lambda_i+\mu_i}$, $p_{i,i-1} = \frac{\mu_i}{\lambda_i+\mu_i}$, resto 0.

(b) Poisson: $\lambda_n = \lambda, \mu_n = 0, \forall n \geq 0$

(c) Yule: $\lambda_n = n\lambda, \mu_n = 0, \forall n \geq 0$

(d) Diagrama de transición.

(e) EEL: $\lambda_0 \pi_0 = \mu_1 \pi_1$, $(\lambda_n + \mu_n)\pi_n = \mu_{n+1} + \pi_{n+1} + \lambda_{n-1}\pi_{n-1}$, $\sum pi = 1$

(f) $\pi_0 = \frac{1}{1+\sum_1^\infty \frac{\lambda_{n-1},\ldots,\lambda_0}{\mu_n,\ldots,\mu_1}}$, $\pi_n = \frac{\lambda_{n-1},\ldots,\lambda_0}{\mu_n,\ldots,\mu_1(1+\sum_1^\infty \frac{\lambda_{n-1},\ldots,\lambda_0}{\mu_n,\ldots,\mu_1})}$

(g) T_i, tiempo de transición $i \to i+1$, $E[T_i] = 1/\lambda_i + \mu_i/\lambda_i E[T_{i-1}]$,
$E[T_0] = 1/\lambda_0, T_0 \sim Exp(\lambda_0)$

(h) $E[T_{ij}] = E[T_i] + E[T_{i+1}] + \ldots + E[T_{j-1}], i < j \in S$

(i) Poisson: $E[T_i] = 1/\lambda \forall i$, $E[T_{ij}] = (j-i)1/\lambda \forall i < j \in S$

(j) Yule: $E[T_i] = 1/(i\lambda)\forall i \geq 1$, $E[T_{ij}] = \sum_{j=i}^{i+n} 1/(j\lambda)\forall i < j \in S$

(k) $Var(T_i) = \frac{1}{\lambda_i(\lambda_i+\mu_i)} + \mu_i/\lambda_i Var(T_{i-1}) + \frac{\mu_i}{\lambda_i+\mu_i}(E[T_{i-1}] + E[T_i])^2$,
$Var(T_0) = 1/\lambda_0^2, T_0 \sim Exp(\lambda_0)$

(l) $Var(T_{ij}) = Var(T_i) + Var(T_{i+1}) + \ldots + Var(T_{j-1}), i < j \in S$

(m) Poisson: $Var(T_i) = 1/\lambda^2$, $Var(T_{ij}) = (i-j)/\lambda^2$

(n) Yule: $Var(T_i) = 1/(i\lambda)^2$, $Var(T_{ij}) = \sum_{j=i}^{i+n} /(j\lambda)^2$

References

[1] Ríos-Insua, S., Mateos-Caballero, A., Bielza, C., Jimenez-Martín, A. (2004), Investigación Operativa. Modelos determinísticos y estocásticos Editorial Centro de Estudios Ramón Areces, S.A.